

# A proof of the linearity conjecture for $k$ -blocking sets in $\text{PG}(n, p^3)$ , $p$ prime

M. Lavrauw \*      L. Storme      G. Van de Voorde \*

## Abstract

In this paper, we show that a small minimal  $k$ -blocking set in  $\text{PG}(n, q^3)$ ,  $q = p^h$ ,  $h \geq 1$ ,  $p$  prime,  $p \geq 7$ , intersecting every  $(n-k)$ -space in 1 (mod  $q$ ) points, is linear. As a corollary, this result shows that all small minimal  $k$ -blocking sets in  $\text{PG}(n, p^3)$ ,  $p$  prime,  $p \geq 7$ , are  $\mathbb{F}_p$ -linear, proving the linearity conjecture (see [7]) in the case  $\text{PG}(n, p^3)$ ,  $p$  prime,  $p \geq 7$ .

## 1 Introduction and preliminaries

Throughout this paper  $q = p^h$ ,  $p$  prime,  $h \geq 1$  and  $\text{PG}(n, q)$  denotes the  $n$ -dimensional projective space over the finite field  $\mathbb{F}_q$  of order  $q$ . A  $k$ -blocking set  $B$  in  $\text{PG}(n, q)$  is a set of points such that any  $(n-k)$ -dimensional subspace intersects  $B$ . A  $k$ -blocking set  $B$  is called *trivial* when a  $k$ -dimensional subspace is contained in  $B$ . If an  $(n-k)$ -dimensional space contains exactly one point of a  $k$ -blocking set  $B$  in  $\text{PG}(n, q)$ , it is called a *tangent  $(n-k)$ -space* to  $B$ . A  $k$ -blocking set  $B$  is called *minimal* when no proper subset of  $B$  is a  $k$ -blocking set. A  $k$ -blocking set  $B$  is called *small* when  $|B| < 3(q^k + 1)/2$ .

Linear blocking sets were first introduced by Lunardon [3] and can be defined in several equivalent ways.

In this paper, we follow the approach described in [1]. In order to define a linear  $k$ -blocking set in this way, we introduce the notion of a Desarguesian spread. Suppose  $q = q_0^t$ , with  $t \geq 1$ . By "field reduction", the points of  $\text{PG}(n, q)$  correspond to  $(t-1)$ -dimensional subspaces of  $\text{PG}((n+1)t-1, q_0)$ , since a point of  $\text{PG}(n, q)$  is a 1-dimensional vector space over  $\mathbb{F}_q$ , and so a  $t$ -dimensional vector space over  $\mathbb{F}_{q_0}$ . In this way, we obtain a partition  $\mathcal{D}$  of the pointset of  $\text{PG}((n+1)t-1, q_0)$  by  $(t-1)$ -dimensional subspaces. In general, a partition of the point set of a projective space by subspaces of a given dimension  $d$  is called a *spread*, or a *d-spread* if we want to specify the dimension. The spread obtained by field reduction is called a *Desarguesian spread*. Note that the Desarguesian spread satisfies the property that each subspace spanned by spread elements is partitioned by spread elements.

Let  $\mathcal{D}$  be the Desarguesian  $(t-1)$ -spread of  $\text{PG}((n+1)t-1, q_0)$ . If  $U$  is a subset of  $\text{PG}((n+1)t-1, q_0)$ , then we define  $\mathcal{B}(U) := \{R \in \mathcal{D} \mid |U \cap R| \neq \emptyset\}$ , and we identify the elements of  $\mathcal{B}(U)$  with the corresponding points of  $\text{PG}(n, q_0^t)$ . If  $U$  is subspace of  $\text{PG}((n+1)t-1, q_0)$ , then we call  $\mathcal{B}(U)$  a *linear set* or an  $\mathbb{F}_{q_0}$ -linear

---

\*This author's research was supported by the Fund for Scientific Research - Flanders (FWO)

set if we want to specify the underlying field. Note that through every point in  $\mathcal{B}(U)$ , there is a subspace  $U'$  such that  $\mathcal{B}(U') = \mathcal{B}(U)$  since the elementwise stabiliser of the Desarguesian spread  $\mathcal{D}$  acts transitively on the points of a spread element of  $\mathcal{D}$ . If  $U$  intersects the elements of  $\mathcal{D}$  in at most a point, i.e.  $|\mathcal{B}(U)|$  is maximal, then we say that  $U$  is *scattered* with respect to  $\mathcal{D}$ ; in this case  $\mathcal{B}(U)$  is called a *scattered linear set*. We denote the element of  $\mathcal{D}$  corresponding to a point  $P$  of  $\text{PG}(n, q_0^t)$  by  $\mathcal{S}(P)$ . If  $U$  is a subset of  $\text{PG}(n, q)$ , then we define  $\mathcal{S}(U) := \{\mathcal{S}(P) \mid P \in U\}$ . Analogously to the correspondence between the points of  $\text{PG}(n, q_0^t)$ , and the elements  $\mathcal{D}$ , we obtain the correspondence between the lines of  $\text{PG}(n, q)$  and the  $(2t-1)$ -dimensional subspaces of  $\text{PG}((n+1)t-1, q_0)$  spanned by two elements of  $\mathcal{D}$ , and in general, we obtain the correspondence between the  $(n-k)$ -spaces of  $\text{PG}(n, q)$  and the  $((n-k+1)t-1)$ -dimensional subspaces of  $\text{PG}((n+1)t-1, q_0)$  spanned by  $n-k+1$  elements of  $\mathcal{D}$ . With this in mind, it is clear that any  $tk$ -dimensional subspace  $U$  of  $\text{PG}(t(n+1)-1, q_0)$  defines a  $k$ -blocking set  $\mathcal{B}(U)$  in  $\text{PG}(n, q)$ . A  $(k)$ -blocking set constructed in this way is called a *linear  $(k)$ -blocking set*, or an  $\mathbb{F}_{q_0}$ -*linear  $(k)$ -blocking set* if we want to specify the underlying field.

By far the most challenging problem concerning blocking sets is the so-called *linearity conjecture*. Since 1998 it has been conjectured by many mathematicians working in the field. The conjecture was explicitly stated in the literature by Sziklai in [7].

(LC) *All small minimal  $k$ -blocking sets in  $\text{PG}(n, q)$  are linear.*

Various instances of the conjecture have been proved; for an overview we refer to [7]. In this paper we prove the linearity conjecture for small minimal  $k$ -blocking sets in  $\text{PG}(n, p^3)$ ,  $p \geq 7$ , as a corollary of the following main theorem:

**Theorem 1.** *A small minimal  $k$ -blocking set in  $\text{PG}(n, q^3)$ ,  $q = p^h$ ,  $p$  prime,  $h \geq 1$ ,  $p \geq 7$ , intersecting every  $(n-k)$ -space in  $1 \pmod{q}$  points is linear.*

## 1.1 Known characterisation results

In this section we mention a few results, that we will rely on in the sequel of this paper. First of all, observe that a subspace intersects a linear set of  $\text{PG}(n, p^h)$  in  $1 \pmod{p}$  or zero points. The following result of Szőnyi and Weiner shows that this property holds for all small minimal blocking sets.

**Result 2.** [8, Theorem 2.7] *If  $B$  is a small minimal  $k$ -blocking set of  $\text{PG}(n, q)$ ,  $p > 2$ , then every subspace intersects  $B$  in  $1 \pmod{p}$  or zero points.*

Result 2 answers the linearity conjecture in the affirmative for  $\text{PG}(n, p)$ . For  $\text{PG}(n, p^2)$ , the linearity conjecture was proved by Weiner (see [9]). For 1-blocking sets in  $\text{PG}(n, q^3)$ , we have the following theorem of Polverino ( $n = 2$ ) and Storme and Weiner ( $n \geq 3$ ).

**Result 3.** [5][6] *A minimal 1-blocking set in  $\text{PG}(n, q^3)$ ,  $q = p^h$ ,  $h \geq 1$ ,  $p$  prime,  $p \geq 7$ ,  $n \geq 2$ , of size at most  $q^3 + q^2 + q + 1$ , is linear.*

In Theorem 8 we show that this implies the linearity conjecture for small minimal 1-blocking sets  $\text{PG}(n, q^3)$ ,  $p \geq 7$ , that intersect every hyperplane in  $1 \pmod{q}$  points.

The following Result by Szőnyi and Weiner gives a sufficient condition for a blocking set to be minimal.

**Result 4.** [8, Lemma 3.1] Let  $B$  be a  $k$ -blocking set of  $\text{PG}(n, q)$ , and suppose that  $|B| \leq 2q^k$ . If each  $(n - k)$ -dimensional subspace of  $\text{PG}(n, q)$  intersects  $B$  in  $1 \pmod{p}$  points, then  $B$  is minimal.

## 1.2 The intersection of a subline and an $\mathbb{F}_q$ -linear set

The possibilities for an  $\mathbb{F}_q$ -linear set of  $\text{PG}(1, q^3)$ , other than the empty set, a point, and the set  $\text{PG}(1, q^3)$  itself are the following: a subline  $\text{PG}(1, q)$  of  $\text{PG}(1, q^3)$ , corresponding to the a line of  $\text{PG}(5, q)$  not contained in an element of  $\mathcal{D}$ ; a set of  $q^2 + 1$  points of  $\text{PG}(1, q^3)$ , corresponding to a plane of  $\text{PG}(5, q)$  that intersects an element of  $\mathcal{D}$  in a line; a set of  $q^2 + q + 1$  points of  $\text{PG}(1, q^3)$ , corresponding to a plane of  $\text{PG}(5, q)$  that is scattered w.r.t.  $\mathcal{D}$ .

The following results describe the possibilities for the intersection of a subline with an  $\mathbb{F}_q$ -linear set in  $\text{PG}(1, q^3)$ , and will play an important role in this paper.

**Result 5.** [2] A subline  $\cong \text{PG}(1, q)$  intersects an  $\mathbb{F}_q$ -linear set of  $\text{PG}(1, q^3)$  in  $0, 1, 2, 3$ , or  $q + 1$  points.

**Result 6.** [4, Lemma 4.4, 4.5, 4.6] Let  $q$  be a square. A subline  $\text{PG}(1, q)$  and a Baer subline  $\text{PG}(1, q\sqrt{q})$  of  $\text{PG}(1, q^3)$  share at most a subline  $\text{PG}(1, \sqrt{q})$ . A Baer subline  $\text{PG}(1, q\sqrt{q})$  and an  $\mathbb{F}_q$ -linear set of  $q^2 + 1$  or  $q^2 + q + 1$  points in  $\text{PG}(1, q^3)$  share at most  $q + \sqrt{q} + 1$  points.

## 2 Some bounds and the case $k = 1$

The Gaussian coefficient  $\begin{bmatrix} n \\ k \end{bmatrix}_q$  denotes the number of  $(k - 1)$ -subspaces in  $\text{PG}(n - 1, q)$ , i.e.,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}.$$

**Lemma 7.** If  $B$  is a subset of  $\text{PG}(n, q^3)$ ,  $q \geq 7$ , intersecting every  $(n - k)$ -space,  $k \geq 1$ , in  $1 \pmod{q}$  points, and  $\pi$  is an  $(n - k + s)$ -space,  $s \leq k$ , then either

$$|B \cap \pi| < q^{3s} + q^{3s-1} + q^{3s-2} + 3q^{3s-3}$$

or

$$|B \cap \pi| > q^{3s+1} - q^{3s-1} - q^{3s-2} - 3q^{3s-3}.$$

*Proof.* Let  $\pi$  be an  $(n - k + s)$ -space of  $\text{PG}(n, q^3)$ , and put  $B_\pi := B \cap \pi$ . Let  $x_i$  denote the number of  $(n - k)$ -spaces of  $\pi$  intersecting  $B_\pi$  in  $i$  points. Counting the number of  $(n - k)$ -spaces, the number of incident pairs  $(P, \sigma)$  with  $P \in B_\pi, P \in \sigma, \sigma$  an  $(n - k)$ -space, and the number of triples  $(P_1, P_2, \sigma)$ , with  $P_1, P_2 \in B_\pi, P_1 \neq P_2, P_1, P_2 \in \sigma, \sigma$  an  $(n - k)$ -space yields:

$$\sum_i x_i = \begin{bmatrix} n - k + s + 1 \\ n - k + 1 \end{bmatrix}_{q^3}, \quad (1)$$

$$\sum_i ix_i = |B_\pi| \begin{bmatrix} n - k + s \\ n - k \end{bmatrix}_{q^3}, \quad (2)$$

$$\sum i(i - 1)x_i = |B_\pi|(|B_\pi| - 1) \begin{bmatrix} n - k + s - 1 \\ n - k - 1 \end{bmatrix}_{q^3}. \quad (3)$$

Since we assume that every  $(n-k)$ -space intersects  $B$  in  $1 \pmod{q}$  points, it follows that every  $(n-k)$ -space of  $\pi$  intersect  $B_\pi$  in  $1 \pmod{q}$  points, and hence  $\sum_i (i-1)(i-1-q)x_i \geq 0$ . Using Equations (1), (2), and (3), this yields that

$$|B_\pi|(|B_\pi|-1)(q^{3n-3k}-1)(q^{3n-3k+3}-1)-(q+1)|B_\pi|(q^{3n-3k+3s}-1)(q^{3n-3k+3}-1) \\ + (q+1)(q^{3n-3k+3s+3}-1)(q^{3n-3k+3s}-1) \geq 0.$$

Putting  $|B_\pi| = q^{3s} + q^{3s-1} + q^{3s-2} + 3q^{3s-3}$  or  $|B_\pi| = q^{3s+1} - q^{3s-1} - q^{3s-2} - 3q^{3s-3}$  in this inequality, with  $q \geq 7$ , gives a contradiction. Hence the statement follows.  $\square$

**Theorem 8.** *A small minimal 1-blocking set in  $\text{PG}(n, q^3)$ ,  $p \geq 7$ , intersecting every hyperplane in  $1 \pmod{q}$  points, is linear.*

*Proof.* Lemma 7 implies that a small minimal 1-blocking set  $B$  in  $\text{PG}(n, q^3)$ , intersecting every hyperplane in  $1 \pmod{q}$  points, has at most  $q^3 + q^2 + q + 3$  points. Since every hyperplane intersects  $B$  in  $1 \pmod{q}$  points, it is easy to see that  $|B| \equiv 1 \pmod{q}$ . This implies that  $|B| \leq q^3 + q^2 + q + 1$ . Result 3 shows that  $B$  is linear.  $\square$

**Corollary 9.** *A small minimal 1-blocking set in  $\text{PG}(n, p^3)$ ,  $p$  prime,  $p \geq 7$ , is  $\mathbb{F}_p$ -linear.*

*Proof.* This follows from Result 2 and Theorem 8.  $\square$

For the remaining of this section, we use the following assumption:

- (B)  $B$  is small minimal  $k$ -blocking set in  $\text{PG}(n, q^3)$ ,  $p \geq 7$ , intersecting every  $(n-k)$ -space in  $1 \pmod{q}$  points.

For convenience let us introduce the following terminology. A *full* line of  $B$  is a line which is contained in  $B$ . An  $(n-k+s)$ -space  $S$ ,  $s < k$ , is called *large* if  $S$  contains more than  $q^{3s+1} - q^{3s-1} - q^{3s-2} - 3q^{3s-3}$  points of  $B$ , and  $S$  is called *small* if it contains less than  $q^{3s} + q^{3s-1} + q^{3s-2} + 3q^{3s-3}$  points of  $B$ .

**Lemma 10.** *Let  $L$  be a line such that  $1 < |B \cap L| < q^3 + 1$ .*

(1) *For all  $i \in \{1, \dots, n-k\}$  there exists an  $i$ -space  $\pi_i$  on  $L$  such that  $B \cap \pi_i = B \cap L$ .*

(2) *Let  $N$  be a line, skew to  $L$ . For all  $j \in \{1, \dots, k-2\}$ , there exists a small  $(n-k+j)$ -space  $\pi_j$  on  $L$ , skew to  $N$ .*

*Proof.* (1) It follows from Result 2 that every subspace on  $L$  intersects  $B \setminus L$  in zero or at least  $p$  points. We proceed by induction on the dimension  $i$ . The statement obviously holds for  $i = 1$ . Suppose there exists an  $i$ -space  $\pi_i$  on  $L$  such that  $\pi_i \cap B = L \cap B$ , with  $i \leq n-k-1$ . If there is no  $(i+1)$ -space intersecting  $B$  only on  $L$ , then the number of points of  $B$  is at least

$$|B \cap L| + p(q^{3(n-i)-3} + q^{3(n-i)-6} + \dots + q^3 + 1),$$

but by Lemma 7  $|B| \leq q^{3k} + q^{3k-1} + q^{3k-2} + 3q^{3k-3}$ . If  $i < n-k-1$  this is a contradiction. If  $i = n-k-1$  then in the above count we may replace the factor  $p$  by a factor  $q$ , using the hypothesis (B), and hence also in this case we get a contradiction. We may conclude that there exists an  $i$ -space  $\pi_i$  on  $L$  such that  $B \cap L = B \cap \pi_i$ ,  $\forall i \in \{1, \dots, n-k\}$ .

(2) Part (1) shows that there is an  $(n - k - 1)$ -space  $\pi_{n-k-1}$  on  $L$ , skew to  $N$ , such that  $B \cap L = B \cap \pi_{n-k-1}$ . If an  $(n - k)$ -space through  $\pi_{n-k-1}$  contains an extra element of  $B$ , it contains at least  $q^2$  extra elements of  $B$ , since a line containing 2 points of  $B$  contains at least  $q + 1$  points of  $B$ . This implies that there is an  $(n - k)$ -space  $\pi_{n-k}$  through  $\pi_{n-k-1}$  with no extra points of  $B$ , and skew to  $N$ .

We proceed by induction on the dimension  $i$ . Lemma 12(1) shows that there are at least  $(q^{3k} - 1)/(q^3 - 1) - q^{3k-5} - 5q^{3k-6} + 1 > q^3 + 1$  small  $(n - k + 1)$ -spaces through  $\pi_{n-k}$  which proves the statement for  $i = 1$ .

Suppose that there exists an  $(n - k + t)$ -space  $\pi_{n-k+t}$  on  $L$ , skew to  $N$ , such that  $B \cap \pi_{n-k+t}$  is a small minimal  $t$ -blocking set of  $\pi_{n-k+t}$ . An  $(n - k + t + 1)$ -space through  $\pi_{n-k+t}$  contains at most  $(q^{3t+4} - 1)(q - 1)$  or more than  $q^{3t+4} - q^{3t+2} - q^{3t+1} - 3q^{3t}$  points of  $B$  (see Lemmas 7 and 13).

Suppose all  $(q^{3k-3t} - 1)(q^3 - 1) - q^3 - 1$   $(n - k + t)$ -spaces through  $\pi_{n-k+t-1}$ , skew to  $N$ , contain more than  $q^{3t+4} - q^{3t+2} - q^{3t+1} - 3q^{3t}$  points of  $B$ . Then the number of points in  $B$  is larger than  $q^{3k} + q^{3k-1} + q^{3k-2} + 3q^{3k-3}$  if  $t \leq k - 3$ , a contradiction.

We may conclude that there exists an  $(n - k + j)$ -space  $\pi_j$  on  $L$  such that  $B \cap \pi_j$  is a small minimal  $i$ -blocking set, skew to  $N$ ,  $\forall j \in \{1, \dots, k - 2\}$ .  $\square$

**Theorem 11.** *A line  $L$  intersects  $B$  in a linear set.*

*Proof.* Note that it is enough to show that  $L$  is contained in a subspace of  $\text{PG}(n, q^3)$  intersecting  $B$  in a linear set. If  $k = 1$ , then  $B$  is linear by Theorem 8, and the statement follows. Let  $k > 1$ , let  $L$  be a line, not contained in  $B$ , intersecting  $B$  in at least two points. It follows from Lemma 10 that there exists an  $(n - k)$ -space  $\pi_L$  such that  $B \cap L = B \cap \pi_L$ . If each of the  $(q^{3k} - 1)/(q^3 - 1)$   $(n - k + 1)$ -spaces through  $\pi_L$  is large, then the number of points in  $B$  is at least

$$\frac{q^{3k} - 1}{q^3 - 1}(q^4 - q^2 - q - 3 - q^3) + q^3 > q^{3k} + q^{3k-1} + q^{3k-2} + 3q^{3k-3},$$

a contradiction. Hence, there is a small  $(n - k + 1)$ -space  $\pi$  through  $L$ , so  $B \cap \pi$  is a small 1-blocking set which is linear by Theorem 8. This concludes the proof.  $\square$

**Lemma 12.** *Let  $\pi$  be an  $(n - k)$ -space of  $\text{PG}(n, q^3)$ ,  $k > 1$ .*

- (1) *If  $B \cap \pi$  is a point, then there are at most  $q^{3k-5} + 4q^{3k-6} - 1$  large  $(n - k + 1)$ -spaces through  $\pi$ .*
- (2) *If  $\pi$  intersects  $B$  in  $(q\sqrt{q} + 1)$ ,  $q^2 + 1$  or  $q^2 + q + 1$  collinear points, then there are at most  $q^{3k-5} + 5q^{3k-6} - 1$  large  $(n - k + 1)$ -spaces through  $\pi$ .*
- (3) *If  $\pi$  intersects  $B$  in  $q + 1$  collinear points, then there are at most  $3q^{3k-6} - q^{3k-7} - 1$  large  $(n - k + 1)$ -spaces through  $\pi$ .*

*Proof.* Suppose there are  $y$  large  $(n - k + 1)$ -spaces through  $\pi$ . Then the number of points in  $B$  is at least

$$y(q^4 - q^2 - q - 3 - |B \cap \pi|) + ((q^{3k} - 1)/(q^3 - 1) - y)x + |B \cap \pi|, \quad (*)$$

where  $x$  depends on the intersection  $B \cap \pi$ .

- (1) In this case,  $x = q^3$  and  $|B \cap \pi| = 1$ . If  $y = q^{3k-5} + 4q^{3k-6}$ , then  $(*)$  is larger than  $q^{3k} + q^{3k-1} + q^{3k-2} + 3q^{3k-3}$ , a contradiction.
- (2) In this case  $x = q^3$  and  $|B \cap \pi| \leq q^2 + q + 1$ . If  $y = q^{3k-5} + 5q^{3k-6}$ , then  $(*)$  is larger than  $q^{3k} + q^{3k-1} + q^{3k-2} + 3q^{3k-3}$ , a contradiction.
- (3) By Result 3 we know that an  $(n - k + 1)$ -space  $\pi'$  through  $\pi$  intersects  $B$  in at least  $q^3 + q^2 + 1$  points, since a  $(q + 1)$ -secant in  $\pi'$  implies that the intersection of  $\pi'$  with  $B$  is non-trivial and not a Baer subplane, hence  $x = q^3 + q^2 - q$ , and  $|B \cap \pi| = q + 1$ . If  $3q^{3k-6} - q^{3k-7}$ , then  $(*)$  is larger than  $q^{3k} + q^{3k-1} + q^{3k-2} + 3q^{3k-3}$ , a contradiction.  $\square$

### 3 The proof of Theorem 1

In the proof of the main theorem, we distinguish two cases. In both cases we need the following two lemmas.

We continue with the following assumption

- (B)  $B$  is small minimal  $k$ -blocking set in  $\text{PG}(n, q^3)$ ,  $p \geq 7$ , intersecting every  $(n - k)$ -space in  $1 \pmod{q}$  points;

and we consider the following properties:

- (H<sub>1</sub>)  $\forall s < k$ : every small minimal  $s$ -blocking set, intersecting every  $(n - s)$ -space in  $1 \pmod{q}$  points, not containing a  $(q\sqrt{q} + 1)$ -secant, is  $\mathbb{F}_q$ -linear;
- (H<sub>2</sub>)  $\forall s < k$ : every small minimal  $s$ -blocking set, intersecting every  $(n - s)$ -space in  $1 \pmod{q}$  points, containing a  $(q\sqrt{q} + 1)$ -secant, is  $\mathbb{F}_{q\sqrt{q}}$ -linear.

**Lemma 13.** *If (H<sub>1</sub>) or (H<sub>2</sub>), and  $S$  is a small  $(n - k + s)$ -space,  $0 < s < k$ , then  $B \cap S$  is a small minimal linear  $s$ -blocking set in  $S$ , and hence  $|B \cap S| \leq (q^{3s+1} - 1)/(q - 1)$ .*

*Proof.* Clearly  $B \cap S$  is an  $s$ -blocking set in  $S$ . Result 2 implies that  $B \cap S$  intersects every  $(n - k + s - s)$ -space of  $S$  in  $1 \pmod{p}$  points, and it follows from Result 4 that  $B \cap S$  is minimal. Now apply (H<sub>1</sub>) or (H<sub>2</sub>).  $\square$

**Lemma 14.** *Suppose (H<sub>1</sub>) or (H<sub>2</sub>). Let  $k > 2$  and let  $\pi_{n-2}$  be an  $(n - 2)$ -space such that  $B \cap \pi_{n-2}$  is a non-trivial small linear  $(k - 2)$ -blocking set, then there are at least  $q^3 - q + 6$  small hyperplanes through  $\pi_{n-2}$ .*

*Proof.* Applying Lemma 13 with  $s = k - 2$ , it follows that  $B \cap \pi_{n-2}$  contains at most  $(q^{3k-5} - 1)/(q - 1)$  points. On the other hand, from Lemmas 7 and 13 with  $s = k - 1$ , we know that a hyperplane intersects  $B$  in at most  $(q^{3k-2} - 1)/(q - 1)$  points or in more than  $q^{3k-2} - q^{3k-4} - q^{3k-5} - 3q^{3k-6}$  points. In the first case, a hyperplane  $H$  intersects  $B$  in at least  $q^{3k-3} + 1 + (q^{3k-3} + q)/(q + 1)$  points, using a result of Szőnyi and Weiner [8, Corollary 3.7] for the  $(k - 1)$ -blocking set  $H \cap B$ . If there are at least  $q - 4$  large hyperplanes, then the number of points in  $B$  is at least

$$(q - 4)(q^{3k-2} - q^{3k-4} - q^{3k-5} - 3q^{3k-6} - \frac{q^{3k-5} - 1}{q - 1}) + (q^3 - q + 5)(q^{3k-3} + 1 + \frac{q^{3k-3} + q}{q + 1} - \frac{q^{3k-5} - 1}{q - 1}) + \frac{q^{3k-5} - 1}{q - 1},$$

which is larger than  $q^{3k} + q^{3k-1} + q^{3k-2} + 3q^{3k-3}$  if  $q \geq 7$ , a contradiction. Hence, there are at most  $q - 5$  large hyperplanes through  $\pi_{n-2}$ .  $\square$

### 3.1 Case 1: there are no $q\sqrt{q} + 1$ -secants

In this subsection, we will use induction on  $k$  to prove that small minimal  $k$ -blocking sets in  $\text{PG}(n, q^3)$ , intersecting every  $(n-k)$ -space in  $1 \pmod{q}$  points and not containing a  $(q\sqrt{q} + 1)$ -secant, are  $\mathbb{F}_q$ -linear. The induction basis is Theorem 8. We continue with assumptions  $(H_1)$  and

$(B_1)$   $B$  is small minimal  $k$ -blocking set in  $\text{PG}(n, q^3)$ ,  $p \geq 7$ , intersecting every  $(n-k)$ -space in  $1 \pmod{q}$  points, not containing a  $(q\sqrt{q} + 1)$ -secant.

**Lemma 15.** *If  $B$  is non-trivial, there exist a point  $P \in B$ , a tangent  $(n-k)$ -space  $\pi$  at the point  $P$  and small  $(n-k+1)$ -spaces  $H_i$ , through  $\pi$ , such that there is a  $(q+1)$ -secant through  $P$  in  $H_i$ ,  $i = 1, \dots, q^{3k-3} - 2q^{3k-4}$ .*

*Proof.* Since  $B$  is non-trivial, there is at least one line  $N$  with  $1 < |N \cap B| < q^3 + 1$ . Lemma 10 shows that there is an  $(n-k)$ -space  $\pi_N$  through  $N$  such that  $B \cap N = B \cap \pi_N$ . It follows from Theorem 11 and Lemma 12 that there is at least one  $(n-k+1)$ -space  $H$  through  $\pi_N$  such that  $H \cap B$  is a small minimal linear 1-blocking set of  $H$ . In this non-trivial small minimal linear 1-blocking set, there are  $(q+1)$ -secants (see Result 3). Let  $M$  be one of those  $(q+1)$ -secants of  $B$ . Again using Lemma 10, we find an  $(n-k)$ -space  $\pi_M$  through  $M$  such that  $B \cap M = B \cap \pi_M$ .

Lemma 12(3) shows that through  $\pi_M$ , there are at least  $\frac{q^{3k}-1}{q^3-1} - 3q^{3k-6} + q^{3k-7} + 1$  small  $(n-k+1)$ -spaces. Let  $P$  be a point of  $M$ . Since in each of these intersections,  $P$  lies on at least  $q^2 - 1$  other  $(q+1)$ -secants, a point  $P$  of  $M$  lies in total on at least  $(q^2 - 1)(\frac{q^{3k}-1}{q^3-1} - 3q^{3k-6} + q^{3k-7} + 1)$  other  $(q+1)$ -secants. Since each of the  $\frac{q^{3k}-1}{q^3-1} - 3q^{3k-6} + q^{3k-7} + 1$  small  $(n-k+1)$ -spaces contains at least  $q^3 + q^2 - q$  points of  $B$  not on  $M$ , and  $|B| < q^{3k} + q^{3k-1} + q^{3k-2} + 3q^{3k-3}$  (see Lemma 7), there are less than  $2q^{3k-2} + 6q^{3k-3}$  points of  $B$  left in the large  $(n-k+1)$ -spaces. Hence,  $P$  lies on less than  $2q^{3k-5} + 6q^{3k-6}$  full lines.

Since  $B$  is minimal,  $P$  lies on a tangent  $(n-k)$ -space  $\pi$ . There are at most  $q^{3k-5} + 4q^{3k-6} - 1$  large  $(n-k+1)$ -spaces through  $\pi$  (Lemma 12(1)). Moreover, since at least  $\frac{q^{3k}-1}{q^3-1} - (q^{3k-5} + 4q^{3k-6} - 1) - (2q^{3k-5} + 6q^{3k-6})$   $(n-k+1)$ -spaces through  $\pi$  contain at least  $q^3 + q^2$  points of  $B$ , and at most  $2q^{3k-5} + 6q^{3k-6}$  of the small  $(n-k+1)$ -spaces through  $\pi$  contain exactly  $q^3 + 1$  points of  $B$ , there are at most  $2q^{3k-2} + 23q^{3k-3}$  points of  $B$  left. Hence,  $P$  lies on at most  $2q^{3k-3} + 23q^{3k-4}$   $(q+1)$ -secants of the large  $(n-k+1)$ -spaces through  $\pi$ . This implies that there are at least  $(q^2 - 1)(\frac{q^{3k}-1}{q^3-1} - 3q^{3k-6} + q^{3k-7} + 1) - (2q^{3k-3} + 23q^{3k-4})$   $(q+1)$ -secants through  $P$  left in small  $(n-k+1)$ -spaces through  $\pi$ . Since in a small  $(n-k+1)$ -space through  $\pi$ , there can lie at most  $q^2 + q + 1$   $(q+1)$ -secants through  $P$ , this implies that there are at least  $q^{3k-3} - 2q^{3k-4}$   $(n-k+1)$ -spaces  $H_i$  through  $\pi$  such that  $P$  lies on a  $(q+1)$ -secant in  $H_i$ .  $\square$

**Lemma 16.** *Let  $\pi$  be an  $(n-k)$ -dimensional tangent space of  $B$  at the point  $P$ . Let  $H_1$  and  $H_2$  be two  $(n-k+1)$ -spaces through  $\pi$  for which  $B \cap H_i = \mathcal{B}(\pi_i)$ , for some 3-space  $\pi_i$  through  $x \in \mathcal{S}(P)$ ,  $\mathcal{B}(x) \cap \pi_i = \{x\}$  ( $i = 1, 2$ ) and  $\mathcal{B}(\pi_i)$  not contained in a line of  $\text{PG}(n, q^3)$ . Then  $\mathcal{B}(\langle \pi_1, \pi_2 \rangle) \subseteq B$ .*

*Proof.* Since  $\langle \mathcal{B}(\pi_i) \rangle$  is not contained in a line of  $\text{PG}(n, q^3)$ , there is at most one element  $Q$  of  $\mathcal{B}(\pi_i)$  such that  $\langle \mathcal{S}(P), Q \rangle$  intersects  $\pi_i$  in a plane. If there is such a plane, then we denote its pointset by  $\mu_i$ , otherwise we put  $\mu_i = \emptyset$ .

Let  $M$  be a line through  $x$  in  $\pi_1 \setminus \mu_1$ , let  $s \neq x$  be a point of  $\pi_2 \setminus \mu_2$ , and note that  $\mathcal{B}(s) \cap \pi_2 = \{s\}$ .

We claim that there is a line  $T$  through  $s$  in  $\pi_2$  and an  $(n-2)$ -space  $\pi_M$  through  $\langle \mathcal{B}(M) \rangle$  such that there are at least 4 points  $t_i \in T, t_i \notin \mu_2$ , such that  $\langle \pi_M, \mathcal{B}(t_i) \rangle$  is small and hence has a linear intersection with  $B$ , with  $B \cap \pi_M = M$  if  $k = 2$  and  $B \cap \pi_M$  is a small minimal  $(k-2)$ -blocking set if  $k > 2$ .

If  $k = 2$ , the existence of  $\pi_M$  follows from Lemma 10(1), and we know from Lemma 12(1) that there are at most  $q+3$  large hyperplanes through  $\pi_M$ . Denote the set of points of  $\mathcal{B}(\pi_2)$ , contained in one of those hyperplanes by  $F$ . Hence, if  $Q$  is a point of  $\mathcal{B}(\pi_2) \setminus F$ ,  $\langle Q, \pi_M \rangle$  is a small hyperplane.

Let  $T_1$  be a line through  $s$  in  $\pi_2 \setminus \mu_2$  and not through  $x$ , and suppose that  $\mathcal{B}(T_1)$  contains at least  $q-3$  points of  $F$ .

Let  $T_2$  be a line in  $\pi_2 \setminus \mu_2$ , through  $s$ , not in  $\langle x, T_1 \rangle$ , not through  $x$ . There are at most  $q+3 - (q-3)$  reguli through  $x$  of  $\mathcal{S}(F)$ , not in  $\langle x, T_1 \rangle$ , and if  $\mu \neq \emptyset$  one element of  $\mathcal{B}(\mu_2)$  is contained  $\mathcal{B}(T_2)$ . Since it is possible that  $\mathcal{B}(s)$  is an element of  $F$ , this gives in total at most 8 points of  $\mathcal{B}(T_2)$  that are contained in  $F$ . This implies, if  $q > 11$ , that at least 5 of the hyperplanes  $\{\langle \pi_M, \mathcal{B}(t) \rangle \mid t \in T_2\}$  are small.

If  $q = 11$ , it is possible that  $\mathcal{B}(T_2)$  contains at least 8 points of  $F$ . If  $T_3$  is a line in  $\pi_2 \setminus \mu_2$ , through  $s$ ,  $\langle x, T_1 \rangle, \langle x, T_2 \rangle$  and not through  $x$ , then there are at least 5 points  $t$  of  $T_3$  such that  $\langle \pi_M, \mathcal{B}(t) \rangle$  is a small hyperplane.

If  $q = 7$  and if  $\mathcal{B}(s) \in \mathcal{B}(F)$ , it is possible that  $\mathcal{B}(T_2), \mathcal{B}(T_3)$ , and  $\mathcal{B}(T_4)$ , with  $T_i$  a line through  $s$  in  $\pi_2 \setminus \mu_2$ , not in  $\langle x, T_j \rangle, j < i$ , not through  $x$ , contain 4 points of  $F$ . A fifth line  $T_5$  through  $s$  in  $\pi_2 \setminus \mu_2$ , not in  $\langle x, T_j \rangle, j < i$ , not through  $x$ , contains at least 5 points  $t$  such that  $\langle \pi_M, \mathcal{B}(t) \rangle$  is a small hyperplane.

If  $k > 2$ , let  $T$  be a line through  $s$  in  $\pi_2 \setminus \mu_2$ , not through  $x$ . It follows from Lemma 10(2) that there is an  $(n-2)$ -space  $\pi_M$  through  $\langle \mathcal{B}(M) \rangle$  such that  $B \cap \pi_M$  is a small minimal  $(k-2)$ -blocking set of  $\text{PG}(n, q^3)$ , skew to  $\mathcal{B}(T)$ . Lemma 14 shows that at most  $q-5$  of the hyperplanes through  $\pi_M$  are large. This implies that at least 5 of the hyperplanes  $\{\langle \pi_M, \mathcal{B}(t) \rangle \mid t \in \mathcal{B}(T)\}$  are small. This proves our claim.

Since  $B \cap \langle \mathcal{B}(t_i), \pi_M \rangle$  is linear, also the intersection of  $\langle \mathcal{B}(t_i), \mathcal{B}(M) \rangle$  with  $B$  is linear, i.e., there exist subspaces  $\tau_i, \tau_i \cap \mathcal{S}(P) = \{x\}$ , such that  $\mathcal{B}(\tau_i) = \langle \mathcal{B}(t_i), \mathcal{B}(M) \rangle \cap B$ . Since  $\tau_i \cap \langle \mathcal{B}(M) \rangle$  and  $M$  are both transversals through  $x$  to the same regulus  $\mathcal{B}(M)$ , they coincide, hence  $M \subseteq \tau_i$ . The same holds for  $\tau_i \cap \langle \mathcal{B}(t_i), \mathcal{S}(P) \rangle$ , implying  $t_i \in \tau_i$ . We conclude that  $\mathcal{B}(\langle M, t_i \rangle) \subseteq \mathcal{B}(\tau_i) \subseteq B$ .

We show that  $\mathcal{B}(\langle M, T \rangle) \subseteq B$ . Let  $L'$  be a line of  $\langle M, T \rangle$ , not intersecting  $M$ . The line  $L'$  intersects the planes  $\langle M, t_i \rangle$  in points  $p_i$  such that  $\mathcal{B}(p_i) \in B$ . Since  $\mathcal{B}(L')$  is a subline intersecting  $B$  in at least 4 points, Result 5 shows that  $\mathcal{B}(L') \subset B$ . Since every point of the space  $\langle M, T \rangle$  lies on such a line  $L'$ ,  $\mathcal{B}(\langle M, T \rangle) \subseteq B$ .

Hence,  $\mathcal{B}(\langle M, s \rangle) \subseteq B$  for all lines  $M$  through  $x$ ,  $M$  in  $\pi_1 \setminus \mu_1$ , and all points  $s \neq x \in \pi_2 \setminus \mu_2$ , so  $\mathcal{B}(\langle \pi_1, \pi_2 \rangle \setminus (\langle \mu_1, \pi_2 \rangle \cup \langle \mu_2, \pi_1 \rangle)) \subseteq B$ . Since every point of  $\langle \mu_1, \pi_2 \rangle \cup \langle \mu_2, \pi_1 \rangle$  lies on a line  $N$  with  $q-1$  points of  $\langle \pi_1, \pi_2 \rangle \setminus (\langle \mu_1, \pi_2 \rangle \cup \langle \mu_2, \pi_1 \rangle)$ , Result 5 shows that  $\mathcal{B}(N) \subset B$ . We conclude that  $\mathcal{B}(\langle \pi_1, \pi_2 \rangle) \subseteq B$ .  $\square$

**Theorem 17.** *The set  $B$  is  $\mathbb{F}_q$ -linear.*

*Proof.* If  $B$  is a  $k$ -space, then  $B$  is  $\mathbb{F}_q$ -linear. If  $B$  is non-trivial small minimal  $k$ -blocking set, Lemma 15 shows that there exists a point  $P$  of  $B$ , a tangent  $(n-k)$ -space  $\pi$  at the point  $P$  and at least  $q^{3k-3} - 2q^{3k-4}$   $(n-k+1)$ -spaces  $H_i$  through



$\pi$  for which  $B \cap H_i$  is small and linear, where  $P$  lies on at least one  $(q+1)$ -secant of  $B \cap H_i$ ,  $i = 1, \dots, s$ ,  $s \geq q^{3k-3} - 2q^{3k-4}$ . Let  $B \cap H_i = \mathcal{B}(\pi_i)$ ,  $i = 1, \dots, s$ , with  $\pi_i$  a 3-dimensional space.

Lemma 16 shows that  $\mathcal{B}(\langle \pi_i, \pi_j \rangle) \subseteq B$ ,  $0 \leq i \neq j \leq s$ .

If  $k = 2$ , the set  $\mathcal{B}(\langle \pi_1, \pi_2 \rangle)$  corresponds to a linear 2-blocking set  $B'$  in  $\text{PG}(n, q^3)$ . Since  $B$  is minimal,  $B = B'$ , and the Theorem is proven.

Let  $k > 2$ . Denote the  $(n - k + 1)$ -spaces through  $\pi$ , different from  $H_i$ , by  $K_j$ ,  $j = 1, \dots, z$ . It follows from Lemma 15 that  $z \leq 2q^{3k-4} + (q^{3k-3} - 1)/(q^3 - 1)$ . There are at least  $(q^{3k-3} - 2q^{3k-4} - 1)/q^3$  different  $(n - k + 2)$ -spaces  $\langle H_1, H_j \rangle$ ,  $1 < j \leq s$ . If all  $(n - k + 2)$ -spaces  $\langle H_1, H_j \rangle$ , contain at least  $5q^2 - 49$  of the spaces  $K_i$ , then  $z \geq (5q^2 - 49)(q^{3k-3} - 2q^{3k-4} - 1)/q^3$ , a contradiction if  $q \geq 7$ . Let  $\langle H_1, H_2 \rangle$  be an  $(n - k + 2)$ -spaces containing less than  $5q^2 - 49$  spaces  $K_i$ .

Suppose by induction that for any  $1 < i < k$ , there is an  $(n - k + i)$ -space  $\langle H_1, H_2, \dots, H_i \rangle$  containing at most  $5q^{3i-4} - 49q^{3i-6}$  of the spaces  $K_i$  such that  $\mathcal{B}(\langle \pi_1, \dots, \pi_i \rangle) \subseteq B$ .

There are at least  $\frac{q^{3k-3} - 2q^{3k-4} - (q^{3i-1} - 1)/(q^3 - 1)}{q^{3i}}$  different  $(n - k + i + 1)$ -spaces  $\langle H_1, H_2, \dots, H_i, H \rangle$ ,  $H \not\subseteq \langle H_1, H_2, \dots, H_i \rangle$ . If all of these contain at least  $5q^{3i-1} - 49q^{3i-3}$  of the spaces  $K_i$ , then

$$z \geq \frac{(5q^{3i-1} - 49q^{3i-3} - 5q^{3i-4} + 49q^{3i-6}) \frac{q^{3k-3} - 2q^{3k-4} - (q^{3i-1} - 1)/(q^3 - 1)}{q^{3i}}}{+ 5q^{3i-4} - 49q^{3i-6}},$$

a contradiction if  $q \geq 7$ . Let  $\langle H_1, \dots, H_{i+1} \rangle$  be an  $(n - k + i + 1)$ -space containing less than  $5q^{3i-1} - 49q^{3i-3}$  spaces  $K_i$ . We still need to prove that  $\mathcal{B}(\langle \pi_1, \dots, \pi_{i+1} \rangle) \subseteq B$ . Since  $\mathcal{B}(\langle \pi_{i+1}, \pi \rangle) \subseteq B$ , with  $\pi$  a 3-space in  $\langle \pi_1, \dots, \pi_i \rangle$  for which  $\mathcal{B}(\pi)$  is not contained in one of the spaces  $K_i$ , there are at most  $5q^{3i-4} - 49q^{3i-6}$  6-dimensional spaces  $\langle \pi_{i+1}, \mu \rangle$  for which  $\mathcal{B}(\langle \pi_{i+1}, \mu \rangle)$  is not necessarily contained in  $B$ , giving rise to at most  $(5q^{3i-4} - 49q^{3i-6})(q^6 + q^5 + q^4)$  points  $t$  for which  $\mathcal{B}(t)$  is not necessarily contained in  $B$ . Let  $u$  be a point of such a space  $\langle \pi_{i+1}, \mu \rangle$ . Suppose that each of the  $(q^{3i+3} - 1)/(q - 1)$  lines through  $u$  in  $\langle \pi_1, \dots, \pi_{i+1} \rangle$  contains at least  $q - 2$  of the points  $t$  for which  $\mathcal{B}(t)$  is not in  $B$ . Then there are at least  $(q - 3)(q^{3i+3} - 1)/(q - 1) + 1 > (5q^{3i-4} - 49q^{3i-6})(q^6 + q^5 + q^4)$  such points  $t$ , if  $q \geq 7$ , a contradiction. Hence, there is a line  $N$  through  $t$  for which for at least 4 points  $v \in N$ ,  $\mathcal{B}(v) \in B$ . Result 5 yields that  $\mathcal{B}(t) \in B$ . This implies that  $\mathcal{B}(\langle \pi_1, \dots, \pi_{i+1} \rangle) \subseteq B$ .

Hence, the space  $\langle H_1, H_2, \dots, H_k \rangle$ , which spans the space  $\text{PG}(n, q^3)$ , is such that  $\mathcal{B}(\langle \pi_1, \dots, \pi_k \rangle) \subseteq B$ . But  $\mathcal{B}(\langle \pi_1, \dots, \pi_k \rangle)$  corresponds to a linear  $k$ -blocking set  $B'$  in  $\text{PG}(n, q^3)$ . Since  $B$  is minimal,  $B = B'$ .  $\square$

**Corollary 18.** *A small minimal  $k$ -blocking set in  $\text{PG}(n, p^3)$ ,  $p$  prime,  $p \geq 7$ , is  $\mathbb{F}_p$ -linear.*

*Proof.* This follows from Results 2 and Theorem 17.  $\square$

### 3.2 Case 2: there are $(q\sqrt{q} + 1)$ -secants to $B$

In this subsection, we will use induction on  $k$  to prove that small minimal  $k$ -blocking sets in  $\text{PG}(n, q^3)$ , intersecting every  $(n - k)$ -space in 1 (mod  $q$ ) points and containing a  $q\sqrt{q} + 1$ -secant, are  $\mathbb{F}_{q\sqrt{q}}$ -linear. The induction basis is Theorem 8. We continue with assumptions  $(H_2)$  and

( $B_2$ )  $B$  is small minimal  $k$ -blocking set in  $\text{PG}(n, q^3)$  intersecting every  $(n-k)$ -space in  $1 \pmod{q}$  points, containing a  $(q\sqrt{q}+1)$ -secant.

In this case,  $\mathcal{S}$  maps  $\text{PG}(n, q^3)$  onto  $\text{PG}(2n+1, q\sqrt{q})$  and the Desarguesian spread consists of lines.

**Lemma 19.** *If  $B$  is non-trivial, there exist a point  $P \in B$ , a tangent  $(n-k)$ -space  $\pi$  at  $P$  and small  $(n-k+1)$ -spaces  $H_i$  through  $\pi$ , such that there is a  $(q\sqrt{q}+1)$ -secant through  $P$  in  $H_i$ ,  $i = 1, \dots, q^{3k-3} - q^{3k-4} - 2\sqrt{q}q^{3k-5}$ .*

*Proof.* There is a  $(q\sqrt{q}+1)$ -secant  $M$ . Lemma 10(1) shows that there is an  $(n-k)$ -space  $\pi_M$  through  $M$  such that  $B \cap M = B \cap \pi_M$ .

Lemma 12(3) shows that there are at least  $\frac{q^{3k}-1}{q^3-1} - q^{3k-5} - 5q^{3k-6} + 1$  small  $(n-k+1)$ -spaces through  $\pi_M$ . Moreover, the intersections of these small  $(n-k+1)$ -spaces with  $B$  are Baer subplanes  $\text{PG}(2, q\sqrt{q})$ , since there is a  $(q\sqrt{q}+1)$ -secant  $M$ . Let  $P$  be a point of  $M \cap B$ .

Since in any of these intersections,  $P$  lies on  $q\sqrt{q}$  other  $(q\sqrt{q}+1)$ -secants, a point  $P$  of  $M \cap B$  lies in total on at least  $q\sqrt{q}(\frac{q^{3k}-1}{q^3-1} - q^{3k-5} - 5q^{3k-6} + 1)$  other  $(q\sqrt{q}+1)$ -secants. Since any of the  $\frac{q^{3k}-1}{q^3-1} - q^{3k-5} - 5q^{3k-6} + 1$  small  $(n-k+1)$ -spaces through  $\pi_M$  contains  $q^3$  points of  $B$  not in  $\pi_M$ , and  $|B| < q^{3k} + q^{3k-1} + q^{3k-2} + 3q^{3k-3}$  (see Lemma 7), there are less than  $q^{3k-1} + 4q^{3k-2}$  points of  $B$  left in the other  $(n-k+1)$ -spaces through  $\pi_M$ . Hence,  $P$  lies on less than  $q^{3k-4} + 4q^{3k-5}$  full lines.

Since  $B$  is minimal, there is a tangent  $(n-k)$ -space  $\pi$  through  $P$ . There are at most  $q^{3k-5} + 4q^{3k-6} - 1$  large  $(n-k+1)$ -spaces through  $\pi$  (Lemma 12(1)). Moreover, since at least  $\frac{q^{3k}-1}{q^3-1} - (q^{3k-5} + 4q^{3k-6} - 1) - (q^{3k-4} + 4q^{3k-5})$  small  $(n-k+1)$ -spaces through  $\pi$  contain  $q^3 + q\sqrt{q} + 1$  points of  $B$ , and at most  $q^{3k-4} + 4q^{3k-5}$  of the small  $(n-k+1)$ -spaces through  $\pi$  contain exactly  $q^3 + 1$  points of  $B$ , there are at most  $q^{3k-1} - q^{3k-2}\sqrt{q} + 4q^{3k-2}$  points of  $B$  left. Hence,  $P$  lies on at most  $(q^{3k-1} - q^{3k-2}\sqrt{q} + 4q^{3k-2}) / (q\sqrt{q} + 1)$  different  $(q\sqrt{q}+1)$ -secants of the large  $(n-k+1)$ -spaces through  $\pi$ . This implies that there are at least  $q\sqrt{q}(\frac{q^{3k}-1}{q^3-1} - q^{3k-5} - 5q^{3k-6} + 1) - (q^{3k-1} - q^{3k-2}\sqrt{q} + 4q^{3k-2}) / (q\sqrt{q} + 1)$  different  $(q\sqrt{q}+1)$ -secants left through  $P$  in small  $(n-k+1)$ -spaces through  $\pi$ . Since in a small  $(n-k+1)$ -space through  $\pi$ , there lie  $q\sqrt{q} + 1$  different  $(q\sqrt{q}+1)$ -secants through  $P$ , this implies that there are certainly at least  $q^{3k-3} - q^{3k-4} - 2\sqrt{q}q^{3k-5}$  small  $(n-k+1)$ -spaces  $H_i$  through  $\pi$  such that  $P$  lies on a  $(q\sqrt{q}+1)$ -secant in  $H_i$ .  $\square$

**Lemma 20.** *Let  $\pi$  be an  $(n-k)$ -dimensional tangent space of  $B$  at the point  $P$ . Let  $H_1$  and  $H_2$  be two  $(n-k+1)$ -spaces through  $\pi$  for which  $B \cap H_i = \mathcal{B}(\pi_i)$ , for some plane  $\pi_i$  through  $x \in \mathcal{S}(P)$ ,  $\mathcal{B}(x) \cap \pi_i = \{x\}$  ( $i = 1, 2$ ) and  $\mathcal{B}(\pi_i)$  not contained in a line of  $\text{PG}(n, q^3)$ . Then  $\mathcal{B}(\langle \pi_1, \pi_2 \rangle) \subseteq B$ .*

*Proof.* Let  $M$  be a line through  $x$  in  $\pi_1$ , let  $s \neq x$  be a point of  $\pi_2$ .

We claim that there is a line  $T$  through  $s$ , not through  $x$ , in  $\pi_2$  and an  $(n-2)$ -space  $\pi_M$  through  $\langle \mathcal{B}(M) \rangle$  such that there are at least  $q\sqrt{q} - q - 2$  points  $t_i \in T$ , such that  $\langle \pi_M, \mathcal{B}(t_i) \rangle$  is small and hence has a linear intersection with  $B$ , with  $B \cap \pi_M = M$  if  $k = 2$  and  $B \cap \pi_M$  is a small minimal  $(k-2)$ -blocking set if  $k > 2$ . From Lemma 12(1), we know that there are at most  $q + 3$  large hyperplanes through  $\pi_M$  if  $k = 2$ , and at most  $q - 5$  if  $k > 2$  (see Lemma 14).

Let  $T$  be a line through  $s$  in  $\pi_2$ , not through  $x$ . The existence of  $\pi_M$  follows from Lemma 10(1) if  $k = 2$ , and Lemma 10(2) if  $k > 2$ . Since  $\mathcal{B}(T)$  contains  $q\sqrt{q} + 1$  spread elements, there are at least  $q\sqrt{q} - q - 2$  points  $t_i \in T$  such that  $\langle \pi_M, \mathcal{B}(t_i) \rangle$  is small. This proves our claim.

Since  $B \cap \langle \mathcal{B}(t_i), \pi_M \rangle$  is linear, also the intersection of  $\langle \mathcal{B}(t_i), \mathcal{B}(M) \rangle$  with  $B$  is linear, i.e., there exist subspaces  $\tau_i$ ,  $\tau_i \cap \mathcal{S}(P) = \{x\}$ , such that  $\mathcal{B}(\tau_i) = \langle \mathcal{B}(t_i), \mathcal{B}(M) \rangle \cap B$ . Since  $\tau_i \cap \langle \mathcal{B}(M) \rangle$  and  $M$  are both transversals through  $x$  to the same regulus  $\mathcal{B}(M)$ , they coincide, hence  $M \subseteq \tau_i$ . The same holds for  $\tau_i \cap \langle \mathcal{B}(t_i), \mathcal{S}(P) \rangle$ , implying  $t_i \in \tau_i$ . We conclude that  $\mathcal{B}(\langle M, t_i \rangle) \subseteq \mathcal{B}(\tau_i) \subseteq B$ .

We show that  $\mathcal{B}(\langle M, T \rangle) \subseteq B$ . Let  $L'$  be a line of  $\langle M, T \rangle$ , not intersecting  $M$ . The line  $L'$  intersects the planes  $\langle M, t_i \rangle$  in points  $p_i$  such that  $\mathcal{B}(p_i) \subseteq B$ . Since  $\mathcal{B}(L')$  is a subline intersecting  $B$  in at least  $q\sqrt{q} - q - 2$  points, Result 6 shows that  $\mathcal{B}(L') \subseteq B$ . Since every point of the space  $\langle M, T \rangle$  lies on such a line  $L'$ ,  $\mathcal{B}(\langle M, T \rangle) \subseteq B$ .

Hence,  $\mathcal{B}(\langle M, s \rangle) \subseteq B$  for all lines  $M$  through  $x$  in  $\pi_2$ , and all points  $s \neq x \in \pi_2$ . We conclude that  $\mathcal{B}(\langle \pi_1, \pi_2 \rangle) \subseteq B$ .  $\square$

**Theorem 21.** *The set  $B$  is  $\mathbb{F}_{q\sqrt{q}}$ -linear.*

*Proof.* Lemma 19 shows that there exists a point  $P$  of  $B$ , a tangent  $(n - k)$ -space  $\pi$  at the point  $P$  and at least  $q^{3k-3} - q^{3k-4} - 2\sqrt{q}q^{3k-5}$   $(n - k + 1)$ -spaces  $H_i$  through  $\pi$  for which  $B \cap H_i$  is a Baer subplane,  $i = 1, \dots, s$ ,  $s \geq q^{3k-3} - q^{3k-4} - 2\sqrt{q}q^{3k-5}$ . Let  $B \cap H_i = \mathcal{B}(\pi_i)$ ,  $i = 1, \dots, s$ , with  $\pi_i$  a plane.

Lemma 20 shows that  $\mathcal{B}(\langle \pi_i, \pi_j \rangle) \subseteq B$ ,  $0 \leq i \neq j \leq s$ .

If  $k = 2$ , the set  $\mathcal{B}(\langle \pi_1, \pi_2 \rangle)$  corresponds to a linear 2-blocking set  $B'$  in  $\text{PG}(n, q^3)$ . Since  $B$  is minimal,  $B = B'$ , and the Theorem is proven.

Let  $k > 2$ . Denote the  $(n - k + 1)$ -spaces trough  $\pi$  different from  $H_i$  by  $K_j$ ,  $j = 1, \dots, z$ . There are at least  $(q^{3k-3} - q^{3k-4} - 2\sqrt{q}q^{3k-5} - 1)/q^3$  different  $(n - k + 2)$ -spaces  $\langle H_1, H_j \rangle$ ,  $1 < j \leq s$ . If all  $(n - k + 2)$ -spaces  $\langle H_1, H_j \rangle$ , contain at least  $2q^2$  of the spaces  $K_i$ , then  $z \geq 2q^2(q^{3k-3} - q^{3k-4} - 2\sqrt{q}q^{3k-5} - 1)/q^3$ , a contradiction if  $q \geq 49$ . Let  $\langle H_1, H_2 \rangle$  be an  $(n - k + 2)$ -spaces containing less than  $2q^2$  spaces  $K_i$ .

Suppose, by induction, that for any  $1 < i < k$ , there is an  $(n - k + i)$ -space  $\langle H_1, H_2, \dots, H_i \rangle$  containing at most  $2q^{3i-4}$  of the spaces  $K_i$ , such that  $\mathcal{B}(\langle \pi_1, \dots, \pi_i \rangle) \subseteq B$ .

There are at least  $\frac{q^{3k-3} - q^{3k-4} - 2\sqrt{q}q^{3k-5} - (q^{3i} - 1)/(q^3 - 1)}{q^{3i}}$  different  $(n - k + i + 1)$ -spaces  $\langle H_1, H_2, \dots, H_i, H \rangle$ ,  $H \not\subseteq \langle H_1, H_2, \dots, H_i \rangle$ .

If all of these contain at least  $2q^{3i-1}$  of the spaces  $K_i$ , then

$$z \geq (2q^{3i-1} - 2q^{3i-4}) \frac{q^{3k-3} - q^{3k-4} - 2\sqrt{q}q^{3k-5} - (q^{3i} - 1)/(q^3 - 1)}{q^{3i}} + 2q^{3i-4},$$

a contradiction if  $q \geq 49$ . Let  $\langle H_1, \dots, H_{i+1} \rangle$  be an  $(n - k + i + 1)$ -space containing less than  $2q^{3i-1}$  spaces  $K_i$ . We still need to prove that  $\mathcal{B}(\pi_1, \dots, \pi_{i+1}) \subseteq B$ .

Since  $\mathcal{B}(\langle \pi_{i+1}, \pi \rangle) \subseteq B$ , with  $\pi$  a plane in  $\langle \pi_1, \dots, \pi_i \rangle$  for which  $\mathcal{B}(\pi)$  is not contained in one of the spaces  $K_i$ , there are at most  $2q^{3i-4}$  4-dimensional spaces  $\langle \pi_{i+1}, \mu \rangle$  for which  $\mathcal{B}(\langle \pi_{i+1}, \mu \rangle)$  is not necessarily contained in  $B$ , giving rise to at most  $2q^{3i-4}(q^6 + q^4\sqrt{q})$  points  $Q_i$  for which  $\mathcal{B}(Q_i)$  is not necessarily in  $B$ . Let  $Q$  be a point of such a space  $\langle \pi_{i+1}, \mu \rangle$ .

There are  $((q\sqrt{q})^{2i+2} - 1)/(q\sqrt{q} - 1)$  lines through  $Q$  in  $\langle \pi_1, \dots, \pi_{i+1} \rangle \cong \text{PG}(2i + 2, q\sqrt{q})$ , and there are at most  $2q^{3i-4}(q^6 + q^4\sqrt{q})$  points  $Q_i$  for which

$\mathcal{B}(Q_i)$  is not necessarily in  $B$ . Suppose all lines through  $Q$  in  $\langle \pi_1, \dots, \pi_{i+1} \rangle \cong \text{PG}(2i+2, q\sqrt{q})$  contain at least  $q\sqrt{q} - q - \sqrt{q}$  points  $Q_i$  for which  $\mathcal{B}(Q_i)$  is not necessarily in  $B$ , then there are at least  $(q\sqrt{q} - q - \sqrt{q} - 1)((q\sqrt{q})^{2i+2} - 1)/(q\sqrt{q} - 1) + 1 > 2q^{3i-4}(q^6 + q^4\sqrt{q})$  points  $Q_i$  for which  $\mathcal{B}(Q_i)$  is not necessarily in  $B$ , a contradiction.

Hence, there is a line  $N$  through  $Q$  in  $\langle \pi_1, \dots, \pi_{i+1} \rangle$  with at most  $q\sqrt{q} - q - \sqrt{q} - 1$  points  $Q_i$  for which  $\mathcal{B}(Q_i)$  is not necessarily contained in  $B$ , hence, for at least  $q + \sqrt{q} + 2$  points  $R \in N$ ,  $\mathcal{B}(R) \in B$ . Result 6 yields that  $\mathcal{B}(Q) \in B$ . This implies that  $\mathcal{B}(\langle \pi_1, \dots, \pi_{i+1} \rangle) \subseteq B$ .

Hence, the space  $\mathcal{B}(\langle H_1, H_2, \dots, H_k \rangle)$  is such that  $\mathcal{B}(\langle \pi_1, \dots, \pi_k \rangle) \subseteq B$ . But  $\mathcal{B}(\langle \pi_1, \dots, \pi_k \rangle)$  corresponds to a linear  $k$ -blocking set  $B'$  in  $\text{PG}(n, q^3)$ . Since  $B$  is minimal,  $B = B'$ .  $\square$

## References

- [1] M. Lavrauw. Scattered spaces with respect to spreads, and eggs in finite projective spaces. PhD Dissertation, Eindhoven University of Technology, Eindhoven, 2001. viii+115 pp.
- [2] M. Lavrauw and G. Van de Voorde. A note on the intersection of  $\mathbb{F}_q$ -linear subsets of  $\text{PG}(1, q^3)$ . Preprint.
- [3] G. Lunardon. Normal spreads. *Geom. Dedicata* **75** (1999), 245–261.
- [4] K. Metsch and L. Storme. Partial  $t$ -spreads in  $\text{PG}(2t+1, q)$ . *Des. Codes Cryptogr.* **18** (1999), 199–216.
- [5] O. Polverino and L. Storme. Small minimal blocking sets in  $\text{PG}(2, q^3)$ . *European J. Combin.* **23** (2002), no. 1, 83–92.
- [6] L. Storme and Zs. Weiner. On 1-blocking sets in  $\text{PG}(n, q)$ ,  $n \geq 3$ . *Des. Codes Cryptogr.* **21** (2000), no. 1-3, 235–251.
- [7] P. Sziklai. On small blocking sets and their linearity. *J. Combin. Theory, Ser. A*, **115** (2008), no. 7, 1167–1182.
- [8] T. Szőnyi and Zs. Weiner. Small blocking sets in higher dimensions. *J. Combin. Theory Ser. A* **95** (2001), no. 1, 88–101.
- [9] Zs. Weiner. Small point sets of  $\text{PG}(n, q)$  intersecting each  $k$ -space in 1 modulo  $\sqrt{q}$  points. *Innov. Incidence Geom.* **1** (2005), 171–180.